Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 4, pp. 703–712, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00095-6

DYNAMICS OF A CONICAL SHELL CONTAINING SUPERSONIC GAS FLOW†

V. M. ALEKSANDROV and S. A. GRISHIN

Moscow

(*Received* 9 April 1993)

Unsteady dynamics of an elastic conical shell with a circular cross-section are considered when the smaller base is clamped and the larger base is free. Inside the shell gas flows supersonically from the smaller end to the larger end. The shell is also transversally loaded. All processes are assumed to axisymmetric. Flutter is investigated and for a stable situation the evolution of the bending of the cone generators under the action of a given load is analysed. The behaviour of the shell is described by the equations of technical theory in mixed form, the influence of the flux being taken into account via a piston model. A non-local approximation to the solution by a special system of orthogonal polynomials is used. Its properties are discussed and numerical results are given.

A review of the literature can be found in [1–6]. The problem of flutter for a conical shell with circular cross-section was discussed in [7, 8].

1. STATEMENT OF THE PROBLEM

Suppose we have a shell whose median surface is a cone of circular cross-section (Fig. 1). The radius of the smaller base is r_0 and of the larger base r_1. The distance between the bases is d. These parameters are uniquely related to the parameters γ, s_0 and s_1 by the formulae

$$r_0 = s_0 \sin\gamma, \quad r_1 = s_1 \sin\gamma, \quad d = (s_1 - s_0)\cos\gamma \tag{1.1}$$

$$\kappa^{-1} = \operatorname{tg}\gamma = (r_1 - r_0)/d, \quad s_0 = r_0\kappa\sqrt{1+\kappa^{-2}}, \quad s_1 = r_1\kappa\sqrt{1+\kappa^{-2}} \tag{1.2}$$

The shell thickness h is assumed to be constant, and the material is elastic, homogeneous, isotropic, with Young's modulus E and Poisson's ratio ν. The density is ρ. We shall describe the stress–strain state of the shell by the equations of the technical theory in mixed from [1]

$$\begin{aligned} &D\nabla^4 w - \nabla_K^2 F - L(w,F) - q = 0 \\ &\nabla^4 F - Eh\nabla_K^2 w + \tfrac{1}{2} EhL(w,w) = 0 \end{aligned} \tag{1.3}$$

where the operators are

$$\begin{aligned} L(w,F) = {} & (s^{-2}F_{\psi\psi} + s^{-1}F_s)w_{ss} + (s^{-2}w_{\psi\psi} + s^{-1}w_s)F_{ss} - \\ & -2(s^{-1}F_{\psi s} - s^{-2}F_\psi)(s^{-1}w_{\psi s} - s^{-2}w_\psi) \end{aligned} \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 4, pp. 123–132, 1994.

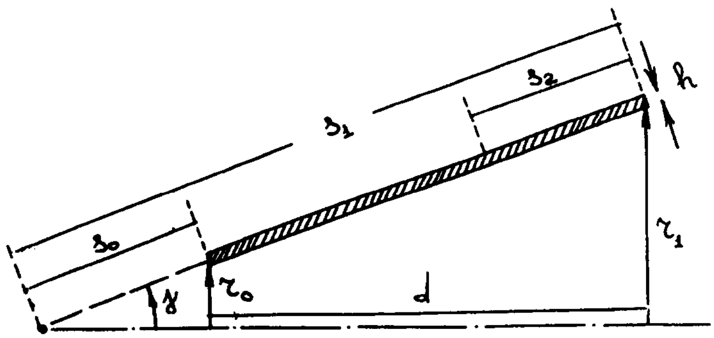

Fig. 1.

$$\nabla^4 = \nabla^2\nabla^2, \quad \nabla^2 = \partial_{ss} + s^{-1}\partial_s + s^{-2}\partial_{\psi\psi}, \quad \nabla_K^2 = s^{-1}\kappa\partial_{ss}$$

and the following notation is used: s is the distance along the median surface meridian from the vertex to a chosen point on the cone, $\psi = \varphi\sin\gamma$, φ is the angular coordinate of that point along the circumference of the cone, w is the normal displacement of the point from the median surface, F is the stress function, $D = \frac{1}{12}Eh^3/(1-\nu^2)$ is the cylindrical rigidity, $\kappa = \operatorname{ctg}\gamma$, ∂_{ss}, ∂_s and $\partial_{\psi\psi}$ are the differential operators with respect to the variables shown by their subscripts, $w_{\psi s}$, w_s, F_{ss} and F_ψ are derivatives with respect to the variables shown by their subscripts, and q is the transverse load whose origin is for the time being unspecified

We change to equations in variations. We assume that a perturbed solution $w + w^0$, $F + F^0$, $q + q^0$ exists near to the fundamental solution w^0, F^0, q^0. Substituting the former into (1.3), using the fact that w^0, F^0, q^0 is a solution, and neglecting terms quadratic in the perturbation, we obtain

$$\begin{gathered} D\nabla^4 w - \nabla_K^2 F - L(w^0, F) - L(w, F^0) - q = 0 \\ \nabla^4 F + Eh\nabla_K^2 w + EhL(w^0, w) = 0 \end{gathered} \tag{1.5}$$

When the fundamental (with index zero) stress–strain state is axially symmetric, we have

$$\begin{gathered} L(w^0, F) = (s^{-2}F_{\psi\psi} + s^{-1}F_s)w_{ss}^0 + F_{ss}s^{-1}w_s^0 \\ L(w, F^0) = s^{-1}F_s^0 w_{ss} + F_{ss}^0(s^{-2}w_{\psi\psi} + s^{-1}w_s) \\ L(w^0, w) = (s^{-2}w_{\psi\psi} + s^{-1}w_s)w_{ss}^0 + w_{ss}s^{-1}w_s^0 \end{gathered} \tag{1.6}$$

When the perturbation is also axially symmetric, we have

$$\begin{gathered} L(w^0, F) = s^{-1}(F_s w_s^0)_s, \quad L(w^0, w) = s^{-1}(w_s w_s^0)_s \\ L(w^0, F) = s^{-1}(F_s^0 w_s)_s \end{gathered} \tag{1.7}$$

As a result (1.5) takes the form

$$\begin{gathered} D\nabla^4 w - \nabla_K^2 F - s^{-1}(F_s w_s^0)_s - s^{-1}(F_s^0 w_s)_s - q = 0 \\ \nabla^4 F - Eh\nabla_K^2 w + Ehs^{-1}(w_s w_s^0)_s = 0 \\ \nabla^4 = s^{-1}\partial_s s\partial_s s^{-1}\partial_s s\partial_s \end{gathered} \tag{1.8}$$

As the principal solution we take the known [1, 2] zero-torque solution

$$(sT_1^0)_s = s\kappa^{-1}q^0, \quad T_2^0 = s\kappa^{-1}q^0 \tag{1.9}$$

where T_1^0 is the longitudinal stress, T_2^0 is the circumferential stress, and q^0 is the constant normal pressure (with $q^0 > 0$ for internal pressure). We require

$$T_1^0(s_1) = 0 \tag{1.10}$$

It is impossible to impose any boundary condition on (1.9) in addition to (1.10). We have to take [3] $w_s^0 = 0$, and so only one of the three expressions in (1.7) is non-zero. Using the fact that with axial symmetry

$$T_2^0 = F_{ss}^0, \quad T_1^0 = s^{-1}F_s^0 \tag{1.11}$$

we calculate

$$s^{-1}(F_s^0 w_s)_s = \tfrac{1}{2}\kappa^{-1}q^0((s - s_1^2 s^{-1})w_{ss} + 2w_s) \tag{1.12}$$

We shall assume that gas is flowing supersonically inside the shell from the narrow end to the wide end (i.e. from left to right in Fig. 1). It exerts a pressure on the wall of the channel, and given the assumption of axial symmetry its mean value (in terms of s lying on $[s_0, \ s_1]$) can be taken to be q^0. Any deviation of the wall causes a pressure perturbation which we will denote by q^1. According to "piston theory" [2, 4, 5] the formula

$$q^1 = -\frac{\rho_1 U}{\sqrt{M^2 - 1}}\left(Uw_s + \frac{M^2 - 2}{M^2 - 1}w_t\right) \tag{1.13}$$

holds.

In (1.13) q^1 and w are functions of both the coordinate s and the time t, M is the Mach number: the ratio of the flow velocity U near the wall to the velocity of sound u in the gas, and ρ_1 is the gas density. All flow parameters are positive, and to simplify the problem they are assumed to be specified in advance and to be constant over s and t.

The inertial forces when the wall moves are taken into account by the term

$$q^2 = -h\rho w_{tt} \tag{1.14}$$

Representing q from (1.8) in the form $q = q^1 + q^2$ we obtain the flutter equations for a shell in a flow. Introducing the additional term $q^3(s, \ t)$ $(q = q^1 + q^2 + q^3)$ we can follow the forced oscillations of the system under the action of a given perturbation q^3.

Combining (1.8)–(1.14) we obtain the basic system of equations:

$$\begin{aligned}
&D\nabla^4 w - \nabla_K^2 F - s^{-1}(F_s^0 w_s)_s + \xi w_s + \eta w_t + \zeta w_{tt} = q^3 \\
&\nabla^4 F + Eh\nabla_K^2 w = 0 \\
&s^{-1}(F_s^0 w_s)_s = \tfrac{1}{2}\kappa^{-1}q^0((s - s_1^2 s^{-1})w_{ss} + 2w_s) \\
&\xi = \rho_1 U^2 (M^2 - 1)^{-1/2} \\
&\eta = \rho_1 U(M^2 - 2)(M^2 - 1)^{-3/2}, \quad \zeta = \rho h
\end{aligned} \tag{1.15}$$

We now consider boundary and initial conditions. The boundary $s=s_1$ is assumed to be free. Then [1] $M_1=0$, $Q_1=0$—the torque and shear stress are zero. In terms of derivatives with respect to w this can be written as

$$s=s_1: \quad w_{ss}+\nu s^{-1}w_s=0, \quad w_{sss}+s^{-1}w_{ss}-s^{-2}w_s=0 \tag{1.16}$$

The left-hand sides of (1.16) are denoted by M_w and Q_w, respectively.

The $s=s_0$ boundary is assumed to be clamped

$$s=s_0: \quad w=0, \quad w_s=0 \tag{1.17}$$

It is natural to require of the function F that at $s=s_1$ the membrane stress T_1 vanishes, from which, by (1.11)

$$s=s_1: \quad F_s=0 \tag{1.18}$$

and also that at $s=s_0$ the displacement along the s axis vanishes.

From the relations of shell theory [1], using (1.17), one can show that $T_2=\nu T_1=0$ $(s=s_0)$. Using (1.11) we arrive at the condition

$$s=s_0: \quad F_{ss}-\nu s^{-1}F_s=0 \tag{1.19}$$

The differential form on the left-hand side of (1.19) only differs from the form earlier denoted by M_w by the sign in front of Poisson's ratio, and so we shall call it M_F and write (1.19) as $M_F=0$ $(s=s_0)$.

The stress function F only participates in physically meaningful shell theory equations through its derivatives, and so without loss of generality it can be fixed at one point

$$s=s_1: \quad F=0 \tag{1.20}$$

Finally, the last condition can be obtained naturally by integrating by parts the functional generated by the leading terms in (1.15)

$$s=s_0: \quad Q_F=F_{sss}+s^{-1}F_{ss}-s^{-2}F_s=0 \tag{1.21}$$

Choosing $s=s_1$ in (1.20) and $s=s_0$ in (1.21) leads to a symmetric system of conditions on w and F in the same sense in which the shell theory equilibrium and compatibility equations are symmetric in the statics-geometry analogy [6].

Initial conditions are not necessary for judging the possibility of flutter, and for dynamics we assume that at the initial instant of time the deflection is zero and the shell does not move

$$t=0: \quad w=0, \quad w_t=0 \tag{1.22}$$

2. METHOD OF SOLUTION

System (1.15) is a linear system of partial differential equations with variable coefficients. We will separate the variables by the Fourier method. We therefore take each of the functions w and F to be a product of a function of time only and a function of the coordinate only, and substitute this factorization into (1.15). From the second equation we find that the time-dependent factors in w and F are identical apart from a multiplicative constant. In the first equation the variables separate provided that η and ζ in (1.15) are connected by a factor independent of s and t. This property was one of the reasons why the flow parameters in

Section 1 were taken to be constant.

For a general configuration the ordinary differential equation for the time-dependent factor, being a linear second-order equation with constant coefficients, has a general solution that is a linear combination of exponentials. Hence flutter-type solutions (of the homogeneous system (1.15)) will be found by separating exponentials of t. If the given shell and flow parameters are such that the system is stable, then it is meaningful to formulate the dynamical problem of the evolution of the shape of the shell under the action of the known loading q^3.

The simplicity of the domain geometry, the fact that the boundary conditions are independent of w and F, and the properties of the leading terms of (1.15) make possible and suggest the use of a non-local coordinate approximation when constructing an approximate solution. We assume that two systems of polynomials $g_k(s)$ and $f_k(s)$ $(k=1, \ldots, \infty)$ with the following properties have been constructed: the g_k satisfying conditions (1.16) and (1.17), and the f_k satisfying conditions (1.8)–(1.21), with

$$(\nabla^4 g_k, g_m) = \delta_{km}, \quad (\nabla^4 f_k, f_m) = \delta_{km}, \quad \forall k,m \tag{2.1}$$

where δ_{km} is the Kronecker delta, and the scalar product

$$(a,b) = \int_{s_0}^{s_1} absds \tag{2.2}$$

Then, representing the solution and the right-hand side by series (where summation in (2.3) and (2.4) is performed with k from ∞)

$$w = \Sigma w_k(t) g_k(s), \quad F = \Sigma F_k(t) f_k(s), \quad q^3 = \Sigma q_k(t) g_k(s) \tag{2.3}$$

and substituting these into (1.15), multiplying the first equation by sg_m $(m=1, \ldots, \infty)$ and the second by sf_j $(j=1, \ldots, \infty)$, integrating from s_0 to s_1 and using (2.1), we arrive at a pair of infinite systems of linear algebraic equations

$$Dw_m - \Sigma(F_k \kappa Y_{mk}^3 + w_k(-\kappa^{-1} q^0 (Y_{mk}^1 - \tfrac{1}{2} Y_{mk}^2) + \xi Y_{mk}^1) + (\eta w_k^\bullet + \zeta w_k^{\bullet\bullet}) Y_{mk}^0) = \Sigma q_k Y_{mk}^0 \tag{2.4}$$

$$F_j + \kappa Eh \Sigma Y_{jk}^4 w_k = 0$$

$$Y_{mk}^0 = (g_m, g_k), \quad Y_{mk}^1 = (g_m, \partial_s g_k), \quad Y_{mk}^2 = (g_m, (s - s_1^2 s^{-1}) \partial_{ss} g_k)$$

$$Y_{mk}^3 = (g_m, s^{-1} \partial_{ss} f_k), \quad Y_{jk}^4 = (f_j, s^{-1} \partial_{ss} g_k) \tag{2.5}$$

The dot denotes differentiation with respect to t.

The expression for F_j in terms of w_k in the second equation of system (2.4) is substituted into the first. It is more convenient henceforth to work in terms of vectors and matrices. Denoting the columns of the coefficients w_k, F_k, q_k from (2.3) by $\mathbf{w}$, $\mathbf{F}$, $\mathbf{q}$, respectively, and taking the Y_{mk}^i from (2.5) to be the components of the matrices $\mathbf{Y}^i$, we rewrite the first system (2.4) in vector from ($\mathbf{I}$ being the unit matrix)

$$\zeta \mathbf{Y}^0 \mathbf{w}^{\bullet\bullet} + \eta \mathbf{Y}^0 \mathbf{w}^\bullet + \mathbf{Y}^5 \mathbf{w} = \mathbf{Y}^0 \mathbf{q} \tag{2.6}$$

$$\mathbf{Y}^5 = D\mathbf{I} + Eh\kappa^2 \mathbf{Y}^3 \mathbf{Y}^4 + (\xi - \kappa^{-1} q^0) \mathbf{Y}^1 + \tfrac{1}{2} \kappa^{-1} q^0 \mathbf{Y}^2$$

The matrix $\mathbf{Y}^0$, being the Gram matrix of a system of linearly-independent functions g_k, is non-degenerate. Multiplying (2.6) on the left by $\zeta^{-1} \mathbf{Y}^{0^{-1}}$ and putting $\mathbf{z} = \mathbf{w}^\bullet$ and $\mathbf{X} = -\zeta^{-1} \mathbf{Y}^{0^{-1}} \mathbf{Y}^5$, we write it as a first-order system in block form

$$\begin{Vmatrix} \mathbf{z} \\ \mathbf{w} \end{Vmatrix}^\bullet = \begin{Vmatrix} -\eta\zeta^{-1}\mathbf{I} & \mathbf{X} \\ \mathbf{I} & \mathbf{0} \end{Vmatrix} \begin{Vmatrix} \mathbf{z} \\ \mathbf{w} \end{Vmatrix} + \begin{Vmatrix} \zeta^{-1}\mathbf{q} \\ 0 \end{Vmatrix} \tag{2.7}$$

We rewrite the initial conditions (1.21) in the form

$$t=0:\quad \mathbf{z}=\mathbf{w}=\mathbf{0} \tag{2.8}$$

We finally truncate series (2.3) after n terms and calculate the right-hand side $\mathbf{q}$ in order to investigate the stability of (2.7) or solve the Cauchy problem (2.7), (2.8) by the standard Runge–Kutta method.

We investigate the bilinear form from the left-hand sides of (2.1). To do this we take two functions $a(s)$, $b(s)$ specified on $[s_0,\ s_1]$, construct the integral $(\nabla^4 a,\ b)$ and reduce the order of the derivatives in the integrand by integrating by parts twice. We then transform the integrated term with the aim of extracting the products of differential forms occurring in boundary conditions (1.16) and (1.17). We finally obtain

$$(\nabla^4 a, b) = (sQ_a b - sM_a b')\Big|_{s_0}^{s_1} + \int_{s_0}^{s_1}(a''b'' + \nu s^{-1}(a''b' + a'b'') + s^{-2}a'b')s\,ds \tag{2.9}$$

Here Q_a and M_a stand for the same forms in a as those of Section 1 are in w, the prime denotes differentiation with respect to s, and we have used the usual notation for the difference in the values of functions at the ends of the interval.

On the lineal of functions satisfying conditions (1.16) and (1.17), the integrated term in (2.9) vanishes and the integral may be treated as a bilinear symmetric functional with arguments a and b. The corresponding quadratic functional is positive, as may be shown by reducing the form in the integrand to a sum of squares, and so the left-hand side of (2.9) defines a new scalar product for such functions

$$\langle a, b\rangle \equiv (\nabla^4 a, b) \tag{2.10}$$

By a similar argument, except that instead of ν we write $-\nu$ throughout, we can establish that the left-hand side of (2.9) defines a scalar product on the lineal of functions satisfying conditions (1.8)–(1.21).

We will now construct a polynomial basis. We consider a polynomial of degree k

$$p_k(s) = \sum_{i=0}^{k} a_i s^i \tag{2.11}$$

Differentiating (2.11), we compute

$$M_p \equiv p_k'' + \nu s^{-1} p_k' = \sum_{i=0}^{k} a_i i(i-1+\nu)s^{i-2}$$

$$Q_p \equiv p_k''' + s^{-1}p_k'' - s^{-2}p_k' = \sum_{i=0}^{k} a_i i^2(i-2)s^{i-3} \tag{2.12}$$

We impose boundary conditions (1.16) and (1.17) on the p_k and express the subordinate coefficients in terms of the leading coefficients. The condition $p_k(s_0)=0$ gives the formula

$$a_0 = -\sum_{i=1}^{k} a_i s_0^i \tag{2.13}$$

The remaining conditions do not contain a_0 and reduce to a system for a_1, a_2, a_3 with matrices and right-hand side of the following form

$$\left.\begin{array}{ccc|l} 1 & 2s_0 & 3s_0^2 & -\sum\limits_{i=4}^{k} a_i i s_0^{i-1} \\ \nu s_1^{-1} & 2(1+\nu) & 3(2+\nu)s_1 & -\sum\limits_{i=4}^{k} a_i i(i-1+\nu)s_1^{i-2} \\ -s_1^{-2} & 0 & 9 & -\sum\limits_{i=4}^{k} a_i i^2(i-2)s_1^{i-3} \end{array}\right. \tag{2.14}$$

The determinant of the matrix (2.14) is equal to

$$6[x^2(1+\nu)-2(1+2\nu)x+3(1+\nu)],\ \ x \equiv s_0/s_1 \tag{2.15}$$

A simple investigation shows that for real $\nu \in [0, \frac{1}{2}]$ it only vanishes when x is complex, so that matrix (2.14) is non-degenerate and the system is always solvable.

In exactly the same way, imposing conditions (1.8)–(1.21) on the p_k, we obtain system (2.14) except that ν is replaced by $-\nu$, and s_0 and s_1 change places. Its matrix is also non-degenerate.

Here it is appropriate to note that the conditions

$$F(s_0)=0,\ \ Q_F(s_1)=0 \tag{2.16}$$

which also make the integrated term in (2.9) vanish, would lead to a system of type (2.14) which would become degenerate for some ratio of ν to s_0/s_1.

It is clear that it is impossible to satisfy four conditions (1.16), (1.17) or (1.18)–(1.21) by a non-trivial polynomial of degree less than four. We therefore begin the sequence of basis polynomials $\{p_k\}$ with a polynomial of degree four. The coefficient of the highest degree term is taken to be 1. From it we calculate the right-hand side of (2.14). Solving the system and using (2.13), we obtain a_1, a_2, a_3, a_0. For the polynomial of degree five we take the coefficients to be 1 for the fifth-degree term and 0 for the fourth-degree term, and obtain the lower degree terms from (2.14) and (2.13). For the polynomial of degree six we take 1 for the sixth-degree term, 0 for the fifth and fourth-degree terms, and take the lower degree terms from (2.14) and (2.13), etc. As a result we construct a polynomial basis $\{p_k\}$ for the corresponding lineal, and using the Gram–Schmidt orthogonalization process with scalar product (2.10) we arrive at a system of functions with the required properties (2.1).

This polynomial basis differs usefully from many others in that the integral (2.9) of the polynomials, which is frequently used in the orthogonalization process, can be calculated explicitly. We represent the polynomial of degree m in the same way as (2.11)

$$h_m(s)=\sum_{j=0}^{m} b_j s^j \tag{2.17}$$

and perform the differentiation, substituting (2.11) and (2.17) into (2.10) and integrating, and obtain

$$\langle p_k, h_m\rangle = \sum_{i=1}^{k}\sum_{j=1}^{m} \frac{ij}{i+j-2}[ij-(1\mp\nu)(i+j-2)]\times$$

$$\times(s_1^{i+j-2}-s_0^{i+j-2})a_i b_j + a_1 b_1 \ln(s_1/s_0) \tag{2.18}$$

where the prime on the summation sign means that terms for which $i=j=1$ and which therefore give a logarithm on being integrated are omitted, and the upper sign is taken when system $\{g_k\}$ is being constructed and the lower sign when $\{f_k\}$ is being constructed. In order to bring the notation into agreement with (2.1), the indices must be shifted: g_1 and f_1 are polynomials of degree four. Graphs of the first few polynomials g_k (for a particular choice of ν and s_0/s_1) are shown in Fig. 2. Note the non-interweaving of the nodes and the rapid

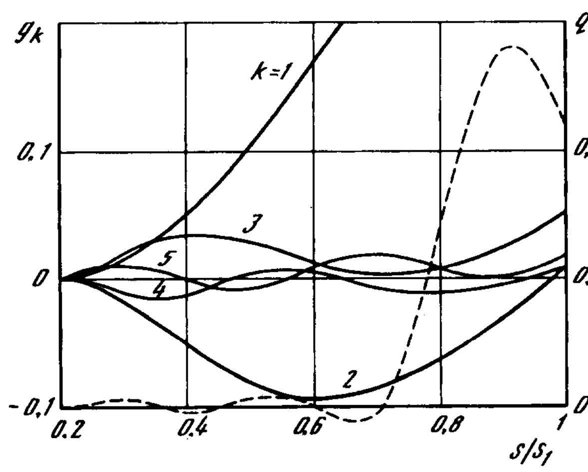

Fig. 2.

amplitude reduction as the number increases.

The basis length n used in actual calculations is completely governed by the properties of the matrix of coefficients of $a_i b_j$ in (2.18). This defect of the polynomial approximation technique has been frequently pointed out [9, 10, 11, 5] and requires careful control of the Gram–Schmidt process: applying re-orthogonalization [12] and monitoring the numerical values of the scalar products $\langle g_k, g_m \rangle$, $\langle f_k, f_m \rangle$. In the series of examples that was considered satisfactory orthogonalization was achieved only for $n \leqslant 8$, which severely restricts the class of admissible loads q^3. It is small consolation that the logic of the shell approach also excludes highly variable loads. In view of this, we consider the extremely awkward case (from the formulational and methodological point of view) of an impulsive load localized along a segment of width s_2 (Fig. 1) near the large end of the cone ($q_* > 0$, $t_0 > 0$ are specified parameters)

$$q^3(s, t) = q^4(s) q^5(t) \tag{2.19}$$

$$q^4 = \begin{cases} 1, & s \in [s_2 - s_1, s_1] \\ 0, & s < s_2 - s_1 \end{cases}, \quad q^5 = \begin{cases} q_*, & 0 < t < t_0 \\ 0, & t \geqslant t_0 \end{cases}$$

For $t \geqslant t_0$ all the q_k in (2.3) are identically equal to zero, while for $0 < t < t_0$ the coefficients q_k do not depend on time, so that for such t we are concerned with the best approximation to the function $q^4(s)$ by the specified system $g_1, \ldots, g_n$ in the sense of the norm induced by (2.2). Minimizing the squared distance from q^4 to the linear hull of the $g_1, \ldots, g_n$ we arrive at a linear system of dimensions $n \times n$

$$\mathbf{Y}^0 \mathbf{q} = \mathbf{R} \tag{2.20}$$

where $\mathbf{R}$ is an n-dimensional vector with components

$$R_i = \int_{s_0}^{s_1} q^4 g_i s ds \tag{2.21}$$

The matrix $\mathbf{Y}^0$ of (2.5) is non-degenerate, while the integrals (2.21) of the characteristic function q^4 in (2.19) are taken explicitly; hence there is no particular difficulty in constructing and solving (2.20). The quality of the approximation of the characteristic function by the basis of available length can be judged for this special case by looking at the dashed curve in Fig. 2.

We now return to system (2.7) and investigate its stability. If n terms are retained in the series (2.3), the matrix (2.7) is of dimensions $2n \times 2n$. Putting $\mathbf{w} = \mathbf{W} e^{\alpha t}$, differentiating with respect to t and substituting into the homogeneous equation (2.6), we arrive at the spectral problem for the matrix $\mathbf{X}$ of dimensions $n \times n$

$$(\mathbf{X} - \lambda\mathbf{I})\mathbf{W} = \mathbf{0}, \quad -\lambda = \alpha(\eta/\zeta + \alpha) \tag{2.22}$$

Finding the eigenvalues λ_i of (2.22), which can be either real or complex-conjugate pairs, we determine $2n$ exponential indices α from n quadratic equations

$$\alpha^2 + \alpha\eta/\zeta + \lambda_i = 0 \tag{2.23}$$

In the calculations the spectral problem was solved both ways—or the matrix $\mathbf{X}$ and for the matrix (2.7)—by the same method: using standard programs from the SSP package the matrix was reduced to Hessenberg form, after which the QR-algorithm was applied. The first way was faster, while the second generally gave better results, which is important when working with a basis of subcritical length.

Two asymptotic checks are possible on the quality of the computations. When $\xi = \eta = q^0 = 0$ the matrix $\mathbf{X}$ is symmetric and positive-definite, the λ_i should be positive, and consequently all the α are pure imaginary. For finite q^0, ξ, η ζ one can bring $\mathbf{X}$ as close to positive-definiteness as is desired by making $D \to \infty$, the α remaining complex and the $\alpha/|\alpha|$ tending to pure imaginary values.

In situations where at least one of the numbers α lay in the right half-plane the system was unstable and there was no point in following its evolution in response to a specific perturbation. If, however, all the α lay in the left half-plane, the inhomogeneous problem was solved. To do this the fourth-order Runge–Kutta method as modified by Gear was applied in the standard implementation of the SSP software package.

3. EXAMPLE

All calculations were performed on an IBM PC 286 in double precision. For the Cauchy problem the display screen showed graphs of the s-dependence of w at different times. Here, as an example, we consider a shell with the following parameters: $E = 78.48 \times 10^9$ Pa, $\nu = 0.25$, $\rho = 1 \times 10^{-9}$ kg m/m^3, $r_0 = 0.1$ m, $r_1 = 0.5$ m, $d = 1.0$ m and $h = 0.001$ m. We take the following averaged flow parameters $u = 1 \times 10^3$ m/s, $\rho_1 = 1 \times 10^{-13}$ kg m/m^3, $q_0 = 19.62 \times 10^4$ Pa, and $U/\sqrt{(1+(\mathrm{tg}\gamma)^2)} = 3 \times 10^3$ m/s. When $n = 6$ six pairs of complex-conjugate eigenvalues are found with almost identical real parts equal to –50, and imaginary parts equal to ±42670, 31220, 22670, 18750, 14960 and 12610, so that the system is stable. The dynamical calculation for a localized impulsive load with $s_2 = 0.2$ m and $t_0 = 1 \times 10^{-6}$ s gives a sensible picture for the evolution of the bending, consistent with the times $t_p = 9.037 \times 10^{-5}$ s and $t_s = 1.565 \times 10^{-4}$ s at which the left boundary is reached by the bulk and shear waves, respectively, in the shell material. The oscillatory process is stable over times of the order of tens of t_p. The role of the parameter q_*, as can also be seen from (1.5) and (2.19), reduces to a renormalization of the ordinate axis for the graphs of $w(s)$.

A completely different picture is observed when $n = 8$. At first the perturbation propagates from right to left, as should be the case, but from a certain time the amplitude of the leading basis function begins to increase rapidly, overwhelming the other terms in the linear combination (2.3). Here the nodes of the graph of $w(s)$ hardly move, creating the impression that the system has lost stability with respect to the dominant harmonic. This is the "saw" that is well known in computational practice [11], which in the case of our short basis acquires a somewhat unfamiliar form. Independently of the unsteady dynamical problem, the investigation of the spectrum of the system matrix in the flutter problem enables us to reject confidently such a numerically unstable situation. Suspicions of numerical instability arise in the analysis of the scalar products that are produced during the orthonormalization process: the error in calculating the leading polynomials increases noticeably. It can also be diagnosed by parametric analysis: the results of the calculation react strongly to insignificant variations in the parameters of the problem such as s_2.

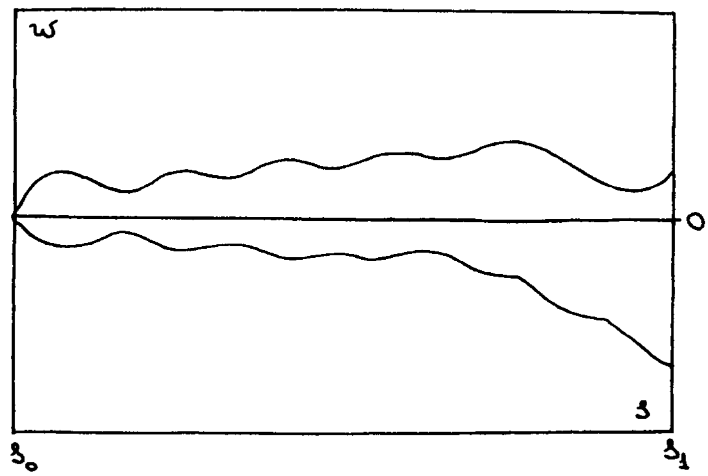

Fig. 3.

If one takes a sufficiently large number of time steps in a stable calculation, one can discover yet another troublesome property of a short basis: an example of the way an envelope of several tens of $w(s)$ curves appears on the display screen is shown in Fig. 3: there are intervals where the domain enclosed by the envelope is noticeably squeezed. These intervals usually coincide with the nodes of the leading polynomial, and the domain of Fig. 3 depends on the value of n.

In spite of these inadequacies, the calculation of an example takes seconds, which is important for multiparameter problems and means that the method is useful in practice. From the numerical results it is very clear that after an impact located near the wide end of the cone, a perturbation wave propagates towards the narrow end where it gives rise to significant strains which can under certain conditions fracture the narrow end of the shell.

REFERENCES

1. GRIGOLYUK E. I. and KABANOV V. V., *The Stability of Shells*. Nauka, Moscow, 1978.
2. VOL'MIR A. S, *The Stability of Deformable Systems*. Nauka, Moscow, 1967.
3. VOROVICH I. I., *Mathematical Problems in the Non-linear Theory of Shallow Shells*. Nauka, Moscow, 1989.
4. GRIGOLYUK E. I., LAMPER R. E. and SHANDAROV I. G., Flutter of panels and shells. In: *Itogi Nauki. Mekh.*, pp. 34–90. VINITI, Moscow, 1965.
5. LAMPER R. E., *Introduction to Flutter Theory*. Mashinostroyeniye, Moscow, 1990.
6. CHERNINA V. S., *Statics of Thin-walled Shells of Revolution*. Nauka, Moscow, 1968.
7. GRIGOLYUK E. I. and MIKHAILOV A. P., Flutter in a three-layered conical shell of circular cross-section. *Dokl. Akad. Nauk SSSR* **163**, 5, 1100–1103, 1965.
8. DITKIN V. V., ORLOV B. A. and PSHENICHNOV G. I., Numerical investigations of flutter in conical shells. *Izv. Ross. Akad. Nauk, MTT* 1, 185–189, 1993.
9. WILKINSON J. H., *The Algebraic Eigenvalue Problem*. Nauka, Moscow, 1970
10. HAMMING R. W., *Numerical Methods for Scientists and Engineers*. Nauka Moscow, 1972.
11. BABENKO K. I., *Foundations of Numerical Analysis*. Nauka, Moscow, 1986.
12. VOYEVODIN V. V., *Numerical Foundations of Linear Algebra*. Nauka, Moscow, 1977.

Translated by R.L.Z.